THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-fifth regular meeting of the Society was held in New York City on Saturday, October 26, 1907, a single morning session sufficing for the unusually brief programme. The attendance included the following twenty-eight members of the Society:

Professor G. A. Bliss, Professor Maxime Bôcher, Professor Joseph Bowden, Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Miss L. D. Cummings, Mr. G. W. Hartwell, Professor E. V. Huntington, Mr. S. A Joffe, Mr. E. H. Koch, Dr. G. H. Ling, Professor E. O. Lovett, Mr. E. B. Lytle, Professor Max Mason, Professor Mansfield Merriman, Dr. R. L. Moore, Mr. H. W. Reddick, Professor L. W. Reid, Mr. F. H. Smith, Professor P. F. Smith, Professor H. D. Thompson, Mr. C. A. Toussaint, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor J. W. Young.

Vice-President Professor P. F. Smith occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. V. R. Aiyar, Gooty, India; Professor P. P. Boyd, Hanover College, Ind.; Dr. Charles Haseman, Indiana University; Professor C. A. Proctor, Dartmouth College; Mr. J. M. Rysgaard, University of North Dakota; Mr. C. A. Toussaint, College of the City of New York. Thirteen applications for admission to membership in the Society were received.

A list of nominations of officers and of the members of the Council was adopted and ordered placed on the official ballot for the annual election at the December meeting.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "A certain class of quartic curves."

(2) Professor R. D. CARMICHAEL: "Geometric properties of quartic curves possessing fourfold symmetry with respect to a point."

(3) Professor OSWALD VEBLEN: "On magic squares."

(4) Professor L. E. DICKSON: "On triple algebras and ternary cubic forms."

(5) Sir G. H. DARWIN: "Further note on Maclaurin's spheroid."

(6) Dr. J. L. COOLIDGE: "The equilong transformations of space."

(7) Professor EDWARD KASNER: "Note on isothermal systems."

(8) Dr. R. L. MOORE: "A note concerning Veblen's axioms for geometry."

(9) Professor JOSEPH BOWDEN: "Proof of a formula in combinations."

Professor Darwin's paper was communicated to the Society through Professor E. B. Van Vleck. In the absence of the authors, the papers of Professor Carmichael, Professor Dickson, Professor Darwin, and Dr. Coolidge were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael studies the general nature of a class of quartic curves whose equation may be represented thus: Set

$$m_i = c_i \sqrt{[(x - a_i)^2 + (y - b_i)^2]} \quad (i = 1, 2, 3),$$

in which $c_i \neq 0$ and the radical is to be taken with the positive sign. Then the equation of the curve studied takes the form

$$(m_1 + m_2 + m_3)(m_1 + m_2 - m_3)(m_1 - m_2 + m_3)(m_1 - m_2 - m_3) = 0,$$

which clears of radicals when the parentheses are removed. A method is also given for the construction of the curve by continuous motion when the c's are commensurable.

2. In a previous communication Professor Carmichael has shown that plane algebraic curves possessing fourfold symmetry with respect to a point divide into two groups. In the present paper he studies the geometric properties of each of these two classes for the case of quartic curves. Special attention is given to the forms of the loci, and the number of singularities in each case is completely determined.

3. Professor Veblen's note shows how it is possible to derive magic squares with p^n elements in a line from the corresponding finite euclidean geometries. The method is a generalization of that of G. Arnoux: Arithmétique graphique. Les espaces arithmétiques hypermagiques, Paris, 1894.

4. Professor Dickson's paper appears in full in the present number of the BULLETIN.

5. In volume 4 of the *Transactions*, Sir George Darwin showed that the form of Maclaurin's ellipsoid could be determined by spherical harmonics to a higher order of approximation than had usually been supposed possible. In the paper referred to he had been unable to determine the angular velocity corresponding to a given ellipticity as far as the cube of that ellipticity. In the present paper he shows that if certain terms previously omitted were retained the defect can be remedied.

6. Scheffers has given the name "equilong" (German, aequilong) to those transformations of the plane which carry a line into a line, keeping invariant the distance between the points of contact of a line with any two envelopes which it touches. These transformations are, in a sense, dual to the conformal ones, and Scheffers has shown that, whereas the latter depend on an arbitrary function of the usual complex variable, the former may be expressed by an arbitrary function of a complex variable of different type. Dr. Coolidge's paper dealt with the corresponding transformations in three dimensions, namely, those which carry a plane into a plane, keeping invariant the distance of the points of contact with any two envelopes which the plane may touch. In this case the duality between conformal and equilong transformations exists in the statement only, for whereas the former group depends on ten arbitrary parameters, the latter involves two arbitrary functions.

7. Professor Kasner's paper appears in full in the present number of the BULLETIN.

8. In volume 5, number 3, pages 343–384 of the *Transactions*, Professor Veblen has exhibited a set of axioms for geometry in terms of point and order. Dr. Moore shows that Axiom II becomes redundant if Axiom V is strengthened by assuming that C is different from B.

9. Professor Bowden gave an elementary proof by mathematical induction of the formula

$$C_r^{m+n} = \sum_{k=1}^{k=r+1} C_{r-k+1}^{m} C_{k-1}^{n}.$$

F. N. COLE,
Secretary.

ON TRIPLE ALGEBRAS AND TERNARY CUBIC FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 26, 1907.)

1. FOR any field F in which there is an irreducible cubic equation $f(\rho) = 0$, the norm of $x + y\rho + z\rho^2$ is a ternary cubic form C which vanishes for no set of values x, y, z in F, other than $x = y = z = 0$. The conditions under which the general ternary form has the last property are here determined for the case of finite fields. One formulation of the result is as follows:

THEOREM. *The necessary and sufficient conditions that a ternary cubic form C shall vanish for no set of values x, y, z in the $GF[p^n]$, $p > 2$, other than $x = y = z = 0$, are that its Hessian shall equal mC, where m is a constant different from zero, and that the binary form obtained from C by setting $z = 0$ shall be irreducible in the field.*

Although I have not hitherto published a proof of this theorem, I have applied it to effect a determination * of all finite triple linear algebras in which multiplication is commutative and distributive, but not necessarily associative, while division is always uniquely possible. I shall here (§ 11) determine these algebras by applying directly the more fundamental conditions from which the preceding theorem is derived.

These ternary cubic forms arise in various other problems; for instance, in the normalization of families of ternary quadratic forms containing three linearly independent forms.

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* *Amer. Math. Monthly*, vol. 13 (1906), pp. 201–205. References are there given to my earlier papers on the subject.

All such ternary cubic forms in a finite field are equivalent under linear transformation in the field (§ 9).

2. Let a ternary form,* with coefficients in the $GF[p^n]$,

$$(1) \qquad C \equiv ax^3 + bx^2y + cx^2z + dxy^2 + exz^2 + fxyz + gy^3 + hy^2z + kyz^2 + lz^3,$$

vanish in the field only for $x = y = z = 0$. Then for any assigned values, not both zero, of y and z in the field, the cubic in x is irreducible; hence it has three roots in the $GF[p^{3n}]$, whose product is $K \equiv -a^{-1}(gy^3 + \cdots + lz^3)$. Now K is irreducible in the $GF[p^n]$. In the $GF[p^{3n}]$ every set of three factors of K, conjugate with respect to the $GF[p^n]$, may be given the form $\gamma(y - \rho z)$, $\gamma^{p^n}(y - \rho^{p^n}z)$, $\gamma^{p^{2n}}(y - \rho^{p^{2n}}z)$, where γ is a root of

$$\gamma^{p^{2n}+p^n+1} = -a^{-1}g.$$

It follows that (1) vanishes for $x = \gamma(y - \rho z)$. In other words, C must have a linear factor, in the $GF[p^{3n}]$,

$$(2) \qquad x - \lambda y - \mu z.$$

For $x = \lambda y + \mu z$, let C become

$$R_0y^3 + R_1y^2z + R_2yz^2 + R_3z^3.$$

Then for y and z arbitrary in the $GF[p^n]$, this sum must vanish for suitably chosen values of λ and μ in the $GF[p^{3n}]$. Hence the four equations $R_i = 0$ must be solvable simultaneously in the $GF[p^{3n}]$.

The conditions $R_i = 0$ are seen to be

$$(3)\ R_0 \equiv a\lambda^3 + b\lambda^2 + d\lambda + g = 0, \quad (4)\ R_3 \equiv a\mu^3 + c\mu^2 + e\mu + l = 0,$$

$$(5)\ R_1 \equiv R_0'\mu + c\lambda^2 + f\lambda + h = 0, \quad (6)\ R_2 \equiv R_3'\lambda + b\mu^2 + f\mu + k = 0,$$

where the accents denote differentiation. If (3) had a root in the $GF[p^n]$, C would vanish for $x = \lambda$, $y = 1$, $z = 0$. Hence (3) and (4) must be irreducible in the $GF[p^n]$. Thus $R_0' \neq 0$, $R_3' \neq 0$.

For λ a root of (3) and for μ defined by (5), we seek the condition under which (2) vanishes for a set of elements x, y, z, not all zero, in the $GF[p^n]$. Eliminating μ between

* To include the cases $p = 2$, $p = 3$, we do not prefix binomial coefficients.

$$x - \lambda y - \mu z = 0$$

and (5), and then eliminating λ^3 by (3), we obtain

$$(7)\quad \lambda^2(3ax + by + cz) + \lambda(2bx + 2dy + fz) + dx + 3gy + hz = 0,$$

Hence such elements x, y, z do not exist if, and only if,

$$(8)\quad \begin{vmatrix} 3a & b & c \\ 2b & 2d & f \\ d & 3g & h \end{vmatrix} \neq 0.$$

THEOREM. *The necessary and sufficient conditions that C shall vanish for no set of values x, y, z in the $GF[p^n]$, other than $x = y = z = 0$, are that C shall have a linear factor* (2) *in the $GF[p^{3n}]$, that* (3) *shall be irreducible in the $GF[p^n]$, and that* (8) *shall hold.*

3. We readily deduce the theorem of § 1 from the preceding theorem. When the conditions of the latter theorem are satisfied, C has three distinct linear factors in the $GF[p^{3n}]$ and hence can be transformed into $\xi\eta\zeta$. The Hessian of the latter is $2\xi\eta\zeta$. In view of the covariance of the Hessian, we conclude that the Hessian of C is of the form mC, m an element $\neq 0$ of the $GF[p^n]$, $p > 2$. Conversely when the Hessian has this property, C has three distinct linear factors.* Each factor is of the form (2), where λ is a root of the irreducible equation (3) and μ is uniquely determined by (5), so that λ and μ belong to the $GF[p^{3n}]$. Further, (8) is satisfied when $m \neq 0$. Indeed, the coefficients of (7) equal $\frac{1}{2}C_{xx}$, C_{xy}, $\frac{1}{2}C_{yy}$, respectively; if they all vanished, the Hessian would vanish.

4. Although the conditions on the coefficients of C may be obtained from the Hessian, we deduce them in convenient form directly from the conditions for the simultaneity of the four equations $R_i = 0$ in the $GF[p^{3n}]$.

For any field we may make $b = 0$ in (1) by an obvious transformation on y and z. Moreover the case $b = 0$ is sufficient for the applications to linear algebras.

* For a direct proof for finite fields, see §§ 5, 6. In the algebraic theory of ternary cubic forms, this property follows from the canonical types (cf. Gordan, *Transactions*, vol. 1, p. 403). We note that if the Hessian of $x^3 + y^3 + z^3 + 6mxyz$ is a multiple of the form, then $m = -\frac{1}{2}\omega$, where $\omega^3 = 1$. Replacing ωz by Z, we obtain the factors $x + y + Z$, $x + \omega y + \omega^2 Z$, $x + \omega^2 y + \omega Z$.

Eliminating μ between (5) and (6) and then eliminating the higher powers of λ by means of (3), we obtain a quadratic function of λ, which must vanish identically, in view of the irreducibility of (3). Hence

$$3aJ = 0, \quad dJ = 0, \quad 3aK + dL = 0, \tag{9}$$

where

$$J = afh - adk - 3aeg + c^2g,$$

$$K = ah^2 - 3agk + cfg - cdh, \tag{10}$$

$$L = ach + 4ade - af^2 - c^2d.$$

But for $b = 0$, (8) becomes

$$6adh - 9afg - 2cd^2 \neq 0. \tag{11}$$

Thus $3a$ and d do not both vanish. Hence

$$J = 0. \tag{12}$$

Eliminating λ between (5) and (6), and dividing by R_3, we obtain the quotient

$$Q \equiv 9a^2d\mu^2 + (3acd + 9a^2h)\mu + c^2d + 3ach - 3ade, \tag{13}$$

and a remainder of degree two, which must vanish identically, in view of the irreducibility of $R_3 = 0$. Hence

$$cL - 3aM = 0, \quad eL + 3aN = 0, \quad eM + cN = 0, \tag{14}$$

where

$$M = 3adl - afk + aeh, \quad N = ak^2 - 3ahl - cdl. \tag{15}$$

We proceed to prove that, if Q is not identically zero, and if conditions (9), (11), (14) are satisfied, and if (3) is irreducible in the $GF[p^n]$, then the four equations $R_i = 0$ are simultaneous in the $GF[p^{3n}]$. Let λ be a root of (3) and μ be defined by (5). In view of the origin of (9), μ satisfies (6). In view of the origin of (14), μ satisfies $QR_3 = 0$. It remains only to show that $Q \neq 0$. First, μ is not an element of the $GF[p^n]$. For, if so, equation (5) and the irreducibility of (3) would give

$$3a\mu + c = 0, \quad f = 0, \quad d\mu + h = 0,$$

and determinant (8) would vanish, having proportional elements in the first and third columns. Next, a mark μ of the $GF[p^{3m}]$, not in the $GF[p^m]$, cannot belong to the $GF[p^{2s}]$. Hence $Q \neq 0$.

5. Finally, let Q be identically zero. The case $p = 3$ is excluded, since then $cd \neq 0$ by (11). Hence $d = h = 0$. Then $fg \neq 0$ by (11). By (9),

$$c^2 = 3ae, \quad cf = 3ak. \tag{16}$$

Then (14) are satisfied, viz., the result of eliminating λ between (5) and (6) now vanishes identically. Removing the factor λ in (5), we now have

$$\mu = (-c\lambda - f)/3a\lambda. \tag{17}$$

Substituting this value in (4), eliminating λ^3 by means of (3), and e by means of (16), we get

$$27a^2gl = c^3g - af^3. \tag{18}$$

In the resulting form C satisfying (16), (18) and

$$b = d = h = 0, \quad fg \neq 0 \qquad (p \neq 3), \tag{19}$$

we replace x by $X - \frac{1}{3}ca^{-1}z$ and obtain

$$aX^3 + gy^3 - \tfrac{1}{27}a^{-1}g^{-1}f^3z^3 + fXyz. \tag{20}$$

Its Hessian is seen to equal $-6f^2C$. Let

$$g = -a\nu, \quad y = -Y, \qquad fz = 3a\nu Z.$$

Then by (20) and (3),

$$C = a(X^3 + \nu Y^3 + \nu^2 Z^3 - 3\nu XYZ), \tag{21}$$

where $\lambda^3 = \nu$ is irreducible in the $GF[p^m]$, whence $p^m = 3m + 1$. After the present change of notation is made, (2) becomes, in view of (17),

$$X + \lambda Y + \lambda^2 Z. \tag{22}$$

This is indeed a factor of (21) for $\lambda^3 = \nu$. Connected with this form (21) is a remarkable non-linear algebra in three units in which division is always uniquely possible.*

* Dickson, *Göttinger Nachrichten*, 1905, p. 359, p. 373.

6. Returning to the case in which Q is not identically zero, we treat first the case in which the modulus exceeds 3. Then we may transform * (1) into a form having $b=c=f=0$. Condititions (9)–(15) then reduce to $dh \neq 0$ and

$$\begin{aligned} &dk+3eg=0, \quad 4de^2+3ak^2-9ahl=0, \\ &eh+3dl=0, \quad 4d^2e+3ah^2-9agk=0. \end{aligned} \tag{23}$$

The second may be derived from the other three. In view of these conditions the Hessian of C reduces to $3deC$. Evidently $e \neq 0$.

Conversely, if the Hessian $3adex^3+\cdots$ of a form C, having $b=c=f=0$, is a non-vanishing multiple of C, then conditions (23) follow. Also $h \neq 0$. For, if $h=0$, conditions (23) give

$$l=0, \quad d=\frac{-3ak^2}{4e^2}, \quad g=\frac{ak^3}{4e^3},$$

and (3) would vanish for $\lambda=-k/e$.

7. We may readily enumerate the resulting forms C. We consider first the case $p>3$, and set $\epsilon=1$ if $p^n=3m+1$, $\epsilon=0$ if $p^n=3m+2$. We set $b=c=0$, thus considering one of p^{2n} coordinate cases. Let first $Q \equiv 0$, so that d, h, e, k all vanish (§ 5). There are $\frac{2}{3}\epsilon(p^n-1)^2$ sets a, g for which $a\lambda^3+g=0$ is irreducible in the $GF[p^n]$. Now f may have any value $\neq 0$; while l is determined by (18). Hence there are

$$\tfrac{2}{3}\epsilon(p^n-1)^3p^{2n} \text{ forms with } Q \equiv 0.$$

For $Q \not\equiv 0$, d and h are not both zero. Let first $d=0$. For each of the $\frac{2}{3}\epsilon(p^n-1)^2$ sets a, g, the coefficients h and e may have any values not zero, f being not zero by (11). Then $f=3egh^{-1}$, $k=\frac{1}{3}g^{-1}h^2$ by $J=K=0$, and M is then zero. Finally, $eL+3aN=0$ determines l. Next, for $d \neq 0$, we make $f=0$ and apply § 6. The number of irreducible cubics $a\lambda^3+d\lambda+g$ with $d \neq 0$ is †

$$\kappa \equiv \tfrac{1}{3}(p^{2n}-1)(p^n-1)-\tfrac{2}{3}\epsilon(p^n-1)^2.$$

* First by $x'=x+\rho y+\sigma z$ we make $b=c=0$. The coefficient of x is a binary quadratic form, so that the term fyz may be deleted.

† BULLETIN, October, 1906, p. 4.

For each set a, d, g, and for any $h \neq 0$, (23) give

$$l = \frac{-eh}{3d}, \quad k = \frac{-3eg}{d}, \quad e(4d^3 + 27ag^2) = -3adh^2,$$

the coefficient of e being the discriminant of the irreducible cubic. In view of b, c, f, we have the factor p^{3n}. Hence there are

$$\tfrac{2}{3}\epsilon(p^n-1)^4p^{2n} + \kappa(p^n-1)p^{2n} \text{ forms with } Q \neq 0.$$

THEOREM.* *The total number of ternary cubic forms in the $GF[p^n]$ which vanish in the field only for $x = y = z = 0$ is*

$$\tfrac{1}{3}(p^{3n}-1)(p^n-1)^2p^{3n}. \tag{24}$$

8. Consider the automorphs of one of our ternary forms C. In view of (3) and (5), we find that $\mu = r\lambda^2 + s\lambda + t$, where $r \neq 0$. Now $C = L_1L_2L_3$, where

$$L_1 = x - \lambda y - \mu z, \quad L_2 = x - \lambda^{p^n}y - \mu^{p^n}z, \quad L_3 = x - \lambda^{p^{2n}}y - \mu^{p^{2n}}z. \tag{25}$$

The determinant D of the coefficients in the L's is a mark $\neq 0$ of the $GF[p^n]$, since $D^{p^n} = D$. Let $L_1' = x' - \lambda y' - \mu z'$, etc. The transformation

$$L_1' = \tau L_1, \quad L_2' = \tau^{p^n}L_2, \quad L_3' = \tau^{p^{2n}}L_3, \tag{26}$$

yields x', y', z' as functions of x, y, z with coefficients in the $GF[p^n]$. Hence there are $p^{2n} + p^n + 1$ automorphs (26) of C. Further,

$$L_1' = L_2, \quad L_2' = L_3, \quad L_3' = L_1 \tag{27}$$

and its square are automorphs of C. Evidently every automorph is generated by (26) and (27).

THEOREM. *The number of automorphs of C is $3(p^{2n}+p^n+1)$.*

9. In view of the order of the general ternary linear homogeneous group in the $GF[p^n]$ and the preceding theorem, it follows that a form C is one of

$$\frac{(p^{3n}-1)(p^{3n}-p^n)(p^{3n}-p^{2n})}{3(p^{2n}+p^n+1)} = \tfrac{1}{3}(p^{2n}-1)(p^n-1)^2p^{3n}$$

conjugates. But this number is the same as (24).

* Another proof results from an enumeration of the distinct products of three linear forms in the $GF[p^{3n}]$ conjugate with respect to the $GF[p^n]$.

THEOREM. *In the $GF[p^n]$, all ternary cubic forms which vanish only for $x = y = z = 0$ are equivalent under linear transformation.*

10. We consider briefly the case* $p = 2$, the Hessian of C being then identically zero. By an obvious transformation we may make $b = c = h = 0$. Then $afg \neq 0$ by (11). Since it remains only to treat the case in which (18) does not vanish identically, we may set $d \neq 0$. Conditions (9)–(14) then reduce to

$$dk = eg, \quad agk = df^2, \quad dl = fk, \quad ef^2 = ak^2,$$

the last being superfluous. We may determine ρ so that $d\rho^2 = a$. We set

$$x = X, \quad y = \rho Y, \quad z = \frac{ga}{fd} Z, \quad \gamma = \frac{g\rho}{d}.$$

Then C has the factor a, which may be made unity by applying a transformation (26). The complementary factor is

$$(28) \quad X^3 + XY^2 + XZ^2 + \gamma XYZ + \gamma Y^3 + \gamma YZ^2 + \gamma^2 Z^3.$$

Multiplying (5) by λ and applying (3), we get

$$g\mu = f\lambda^2.$$

Let $\tau = \rho\lambda$. Then (3) becomes

$$(29) \quad \tau^3 + \tau + \gamma = 0.$$

The factor (2) of C is seen to equal

$$(30) \quad X + \tau Y + \tau^2 Z.$$

By a preliminary transformation on x and y, the irreducible cubic (29) may be transformed into any particular one.

THEOREM. *In the $GF[2^n]$, every ternary cubic form which vanishes only for $x = y = z = 0$ may be transformed into* (28), *where γ is a particular mark for which* (29) *is irreducible.*

11. We proceed to determine all finite triple linear algebras in which multiplication is commutative and distributive, but

* For $p = 3$, I have determined canonical types of all ternary cubic forms. The results are to appear shortly in the *American Journal*.

not necessarily associative, while division is always uniquely possible. We may assume (*Göttinger Nachrichten*, l. c.) that the units are $1, i, j$, where

$$(31) \qquad i^2 = j, \quad ij = ji = g - di, \quad j^2 = h + \delta i + Dj,$$

$x^3 + dx - g$ being irreducible in the $GF[p^n]$. In

$$(x + yi + zj)(\xi + \eta i + \zeta j) = P + Qi + Rj,$$

the determinant of the coefficients of ξ, η, ζ in P, Q, R is of the form (1) with

$$(32) \qquad \begin{aligned} &a = 1, \quad b = 0, \quad c = D - d, \quad e = -h - dD, \\ &f = -\delta - 2g, \quad k = -gD, \quad l = \delta g + dh, \end{aligned}$$

the coefficients g, d, h being the same in the two forms. Let λ be a root, in the $GF[p^{3n}]$, of (3), viz.,

$$(3') \qquad \lambda^3 + d\lambda + g = 0.$$

Thus $-\lambda$ plays a rôle analogous to the unit i of the algebra. We may regard $\lambda, d, g, h, \delta, D$ to be of dimensions 1, 2, 3, 4, 3, 2 respectively. Hence we shall set *

$$(33) \qquad h = \epsilon d^2, \quad \delta = \tau g, \quad D = \kappa d,$$

ϵ, τ, κ being of dimension zero. Then, by (5),

$$-\mu = [(\kappa - 1)d\lambda^2 - (\tau + 2)g\lambda + \epsilon d^2] \div (3\lambda^2 + d).$$

The numerator is of dimension 4, the denominator 2. Hence

$$(5') \qquad -\mu = \rho\lambda^2 + \sigma d,$$

where ρ, σ are of dimension zero. Equating the two values and reducing by (3'), we obtain

$$(34) \qquad \kappa - 1 = 3\sigma - 2\rho, \quad \tau + 2 = 3\rho, \quad \epsilon = \sigma.$$

Next, (6) becomes

$$[3\mu^2 + 2(\kappa - 1)d\mu - (\kappa + \epsilon)d^2]\lambda - (\tau + 2)g\mu - \kappa dg = 0.$$

Eliminating μ by (5'), reducing by (3'), and applying (34), we get

* The case $d = 0$ may be avoided by a transformation of units.

$$4\rho\sigma - \rho^2 - 3\sigma^2 - 4\sigma + 2\rho - 1 = 0, \tag{35}$$
$$(\rho - 1)(\rho - 1 - 3\sigma) = 0,$$

the coefficient of λ^2 being zero. For $\rho = 1$, $\sigma = 0$, and the algebra is a field. For $\rho = 1 + 3\sigma$, (35_1) is satisfied; then $\kappa = -\rho$. Substituting (5′) in (4) and reducing by (3′), we find that the coefficients of λ^2 and λ vanish, and that the constant term is

$$-\sigma^2(\sigma + 1)(4d^3 + 27g^2) = 0.$$

But the second factor is not zero in view of the irreducibility of (3′). For $\sigma = 0$, the algebra is a field. For $\sigma = -1$, $\rho = -2$, and we obtain the non-field algebra

$$i^2 = j, \quad ij = ji = g - di, \quad j^2 = -d^2 - 8gi + 2dj. \tag{36}$$

THE UNIVERSITY OF CHICAGO,
September, 1907.

ISOTHERMAL SYSTEMS IN DYNAMICS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, October 26, 1907.)

CONSIDER any simply infinite system of plane curves defined by its differential equation

$$y' = f(x, y). \tag{1}$$

The ∞^2 isogonal trajectories satisfy the equation *

$$y'' = (F_x + y' F_y)(1 + y'^2), \tag{2}$$

where

$$F = \tan^{-1} f.$$

The theorem of Cesàro-Scheffers states that the trajectories passing through a given point have circles of curvature forming a pencil. We inquire whether any hyperosculating circles exist.

* Primes are employed to denote derivatives with respect to x, and literal subscripts to denote partial derivatives.

The discussion is valid for the general type

$$y'' = (\psi - y'\phi)(1 + y'^2) \tag{3}$$

which arises in dynamics * and includes (2) as a special case. Denoting the radius of curvature by r and the arc by s, we find, for the curves (3)

$$r = \frac{\sqrt{1 + y'^2}}{\psi - y'\phi}, \tag{4}$$

$$\frac{dr}{ds} = \frac{(\phi_y - \phi\psi)y'^2 + (\phi_x - \psi_y + \psi^2 - \phi^2)y' - \psi_x + \phi\psi}{(\psi - y'\phi)^2}. \tag{5}$$

The condition for four-point contact is the vanishing of (5). Hence for a doubly infinite system of type (3) there are at each point two curves which have contact of the third order with their circles of curvature (at that point). The directions of these curves are given by

$$(\phi_y - \phi\psi)y'^2 + (\phi_x - \psi_y + \psi^2 - \phi^2)y' - \psi_x + \phi\psi = 0. \tag{6}$$

The condition that these directions be orthogonal is

$$\phi_y - \psi_x = 0. \tag{7}$$

Applying this to the isogonal type (2), we find

$$F_{xx} + F_{yy} = 0, \tag{8}$$

which is recognized as the condition that (1) be isothermal. We have thus obtained a general property of isogonals and a concrete test for an isothermal system.

The isogonal trajectories of any given simply infinite system of curves are such that through each point there pass two with hyperosculating circles at that point.

These circles intersect orthogonally, for every point of the plane, when and only when the given system is isothermal.

The existence of such orthogonal circles is one of the essential properties of dynamical trajectories produced by conservative forces. When condition (7) is fulfilled, the force whose components are ϕ, ψ is conservative, and the totality of ∞^3 possible

* In the study of certain loci termed velocity curves by the author.

trajectories may be decomposed into ∞^1 "natural families," one for each value of the energy constant. Such a family is defined then by an equation of type (3) with condition (7) fulfilled.

The isogonals of a simple system of curves form a "natural family" when and only when the given system is isothermal.

The explicit relation may be obtained as follows: If the work function defining the field is denoted by W, and the constant of energy is taken to be zero, the natural family is represented by

$$y'' = \tfrac{1}{2}(W_y - y' W_x)(1 + y'^2)/W. \tag{9}$$

If this is to be identical with the isogonal system (2), we must have

$$F_x = \tfrac{1}{2} W_y/W, \qquad F_y = -\tfrac{1}{2} W_x/W. \tag{10}$$

These are seen to be the conditions that the functions F and $\log \sqrt{W}$ be conjugate harmonic. It follows that the base system of the isogonals is isothermal, and that the corresponding field of force is defined by a work function of the form

$$W = e^{2H}, \tag{11}$$

where H is the conjugate harmonic of F, that is, $F + iH$ is a function of $x + iy$. Our result may now be recast as follows:

A natural family of dynamical trajectories can be identified with a system of isogonal trajectories when and only when the work function is the exponential of an harmonic function.

The field of force associated with a given isothermal system is determined up to a constant factor. The lines of force are the orthogonals of the equipotential lines $W = \text{const.}$ and are therefore represented by $F = \text{const.}$; they are seen to be the slope lines of the given system, *i. e.*, the lines joining the points where the curves (1) have parallel tangents. If only the lines of force (themselves necessarily isothermal) are given, the function F may be replaced by any function of the form $c_1 F + c_2$, and the work function is determined except for a constant factor and a constant exponent.

The most immediate application of our results is to central forces varying according to the nth power of the radius vector. This is in fact the only case in which a central force can be of type (11). Using polar coordinates we have

$$W = \rho^{n+1}, \quad H = \tfrac{1}{2}(n+1)\log \rho, \quad F = -\tfrac{1}{2}(n+1)\theta. \tag{12}$$

Among the orbits produced by a central force varying as the nth power of the radius vector are included the isogonal trajectories of the curves

(13) $$y' = -\tan \tfrac{1}{2}(n+1)\theta.$$

This construction yields in the case $n = -2$ (Newtonian law) parabolas with focus at the origin; in the case $n = 1$ (elastic law) equilateral hyperbolas with center at the origin; in the case $n = -5$ the circles through the origin; and in the case $n = -3$ equiangular spirals with pole at the origin.

COLUMBIA UNIVERSITY.

ON THE EQUATIONS OF QUARTIC SURFACES IN TERMS OF QUADRATIC FORMS.

BY DR. C. H. SISAM.

(Read before the American Mathematical Society, September 5, 1907.)

THE quartic surfaces whose equations are of the form

$$\phi_2(A, B, C, D) = 0,$$

where ϕ_2, A, B, C and D are quaternary quadratic forms, were the subject of a paper by H. Durrande in the *Nouvelles Annales*.* By counting the number of constants involved, Durrande concluded that the most general quartic surface could be represented by an equation of the above form. He recognized, however, that his reasoning was not rigorous.

It will here be shown that the coefficients of the quartic surface determined by this equation are not independent, but are subject to a single condition. It will also be shown that *the equation of a general quartic surface can be written in the form*

$$\theta_2(A, B, C, D, E) = 0,$$

where θ_2 is a quinary quadratic form.

Let $\phi_2 = 0$ be reduced to the form

(1) $$A^2 + B^2 + C^2 + D^2 = 0,$$

where

$$A \equiv \Sigma a_{ij} x_i x_j \qquad (i \leqq j \leqq 4)$$

* Durrande, *Nouvelles Annales*, ser. 2, vol. 9, p. 410.

with similar expressions for B, C, and D. Let also

$$\Sigma F_{ijkl}x_ix_jx_kx_l = 0 \qquad (i \leqq j \leqq k \leqq l \leqq 4)$$

be the equation of the same surface in the ordinary form.

Equating the coefficients of like terms in the two equations of the surface, we obtain the following thirty-five equations:

$$\Sigma a_{ii}^2 = F_{iiii} \qquad (i = 1, 2, 3, 4),$$

$$\Sigma 2(a_{ii}a_{ij}) = F_{iiij} \quad (i, j = 1, 2, 3, 4; i \neq j),$$

$$\Sigma(2a_{ij}a_{jj} + a_{ij}^2) = F_{iijj} \quad (i, j = 1, 2, 3, 4; i \neq j),$$

$$\Sigma(2a_{ii}a_{jk} + 2a_{ij}a_{ik}) = F_{iijk} \quad (i, j, k = 1, 2, 3, 4; i \neq j \neq k),$$

$$\Sigma 2(a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23}) = F_{1234},$$

the summation extending over the quantities b, c, and d.

The necessary and sufficient condition that (1) be the equation of a general quartic surface is that these quantities F_{ijkl} be independent functions of a_{ij}, b_{ij}, etc. They will be independent functions of these quantities if, and only if, at least one determinant of order thirty-five of the Jacobian matrix of thirty-five rows and forty columns

$$\begin{vmatrix} \partial F_{1111}/\partial a_{11} & \partial F_{1111}/\partial a_{12} & \cdots & \partial F_{1111}/\partial d_{34} \\ \partial F_{1112}/\partial a_{11} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \\ \partial F_{1234}/\partial a_{11} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \partial F_{1234}/\partial d_{34} \end{vmatrix}$$

be different from zero.

If any of these determinants is different from zero, it will be impossible to find thirty-five quantities, $y_1, y_2, \cdots, y_{35}$, satisfying simultaneously the forty equations

$$\frac{\partial F_{1111}}{\partial a_{11}} \cdot y_1 + \frac{\partial F_{1112}}{\partial a_{11}} \cdot y_2 + \cdots + \frac{\partial F_{1234}}{\partial a_{11}} \cdot y_{35} = 0,$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\frac{\partial F_{1111}}{\partial d_{34}} \cdot y_1 + \frac{\partial F_{1112}}{\partial d_{34}} y_2 + \cdots + \frac{\partial F_{1234}}{\partial d_{34}} \cdot y_{35} = 0.$$

Consider, now, the system of quadrics

$$\alpha A + \beta B + \gamma C + \delta D = 0.$$

In this system there are, in general, ten * which break up into pairs of planes. Let the coordinate planes $x_1 = 0$ and $x_2 = 0$ be taken to coincide with one of these pairs of planes and $x_3 = 0$, $x_4 = 0$ with another pair. We may then, in the above forty equations, put $c_{12} = 1$ and the remaining $c_{ij} = 0$, also $d_{34} = 1$ and the remaining $d_{ij} = 0$.

It is now necessary that $y_{26} = y_{31} = y_{27} = y_{34} = y_{22} = y_5 = y_{25} = y_8 = y_{28} = y_{29} = y_{13} = y_{32} = y_{16} = y_{17} = y_{35} = y_{23} = y_{30} = y_{24} = y_{33} = 0$. Twenty-four of the equations are therefore satisfied. In order that the other sixteen be satisfied by quantities y not all zero, it is necessary and sufficient that the determinant

$$\begin{vmatrix}
a_{11}b_{11} & & & & & & & & & & & & & & & \\
 & a_{22}b_{22} & & & & & & & & & & & & & & \\
 & & a_{33}b_{33} & & & & & & & & & & & & & \\
 & & & a_{44}b_{44} & & & & & & & & & & & & \\
a_{13}b_{13} & & & & a_{11}b_{11} & & & & & & & & & & & \\
a_{14}b_{14} & & & & & a_{11}b_{11} & & & & & & & & & & \\
 & a_{23}b_{23} & & & & & a_{22}b_{22} & & & & & & & & & \\
 & a_{24}b_{24} & & & & & & a_{22}b_{22} & & & & & & & & \\
 & & a_{13}b_{13} & & a_{33}b_{33} & & & & & & & & & & & \\
 & & a_{23}b_{23} & & & & a_{33}b_{33} & & & & & & & & & \\
 & & & a_{14}b_{14} & & a_{44}b_{44} & & & & & & & & & & \\
 & & & a_{24}b_{24} & & & & a_{44}b_{44} & & & & & & & & \\
a_{33}b_{33} & & a_{11}b_{11} & & a_{13}b_{13} & & & & & & & & & & & \\
a_{44}b_{44} & & & a_{11}b_{11} & & a_{14}b_{14} & & & & & & & & & & \\
 & a_{33}b_{33} & a_{22}b_{22} & & & & a_{23}b_{23} & & & & & & & & & \\
 & a_{44}b_{44} & & a_{22}b_{22} & & & & a_{24}b_{24} & & & & & & & &
\end{vmatrix}$$

be equal to zero. Denote the columns of this determinant in order by (1), (2), etc. Replace the first column by $b_{11}(1) - a_{11}(2) + b_{22}(3) - a_{22}(4) + b_{33}(5) - a_{33}(6) + b_{44}(7) - a_{44}(8) + b_{13}(9) - a_{13}(10) + b_{14}(11) - a_{14}(12) + b_{23}(13) - a_{23}(14) + b_{24}(15) - a_{24}(16)$. The first column so obtained vanishes identically. The determinant therefore vanishes identically and the F_{ijkl} are not independent. The surface $\phi_2(A, B, C, D) = 0$ is not, therefore, the most general quartic.

In order that the equation of a quartic may be put in the above form, only a single condition must be satisfied by the coefficients. For if we omit F_{1234}, and consider the remaining F_{ijkl} as functions of the a_{ij}, etc., the Jacobian matrix does not vanish iden-

* Reye, *Crelle*, vol. 82, p. 76; vol. 86, p. 89.

tically. This may be seen by substituting the particular values $a_{11} = b_{22} = c_{33} = d_{44} = 1$, the remaining a_{ij}, etc., being zero. Hence these quantities F_{ijkl} are independent.

Finally, consider the quartic

$$\theta_2(A, B, C, D, E) = 0.$$

Let this equation be reduced to

$$A^2 + B^2 + C^2 + D^2 + E^2 = 0.$$

Equate the coefficients of the terms of this equation to the corresponding terms of

$$\Sigma k_{ijkl}\, x_i x_j x_k x_l = 0 \qquad (i \leqq j \leqq k \leqq l \leqq 4)$$

and determine the jacobian matrix as before. That the determinants of this matrix do not all vanish identically is seen by taking for A, B, etc., the particular expressions

$$A \equiv x_1^2, \quad B \equiv x_2^2, \quad C \equiv x_3^2, \quad D \equiv x_4^2, \quad E \equiv x_1 x_2.$$

Since these determinants do not vanish identically the k_{ijkl} are independent. Hence *the equation of an arbitrary quartic surface can be put into the form*

$$A^2 + B^2 + C^2 + D^2 + E^2 = 0.$$

URBANA, ILL.,
September, 1907.

SYMBOLIC LOGIC.

L'Algèbre de la Logique. Par LOUIS COUTURAT. Collection Scientia, No. 24. Gauthier-Villars, Paris, 1905. 100 pp.

Symbolic Logic and its Applications. By HUGH MACCOLL. Longmans, Green, and Co., London, 1906. xi + 141 pp.

The Development of Symbolic Logic; a Critical-Historical Study of the Logical Calculus. By A. T. SHEARMAN. Williams and Norgate, London, 1906. xi + 242 pp.

SYMBOLIC logic is in the interesting though somewhat precarious state of being little known, less used, and much scorned by the majority of mathematicians and philosophers, for whom it might supposedly offer a region of intimate contact and

mutual admiration. Meanwhile it has its own ardent supporters whose proselytism is at times almost as fanatical and extravagant as it is unavailing. One is tempted to draw a parallel with the subject of quaternions, which was offered insistently but ineffectively at bargain prices to both mathematicians and physicists until the nearly simultaneous death of Tait and Joly caused a temporary and perhaps permanent lull in the propaganda.

How far the analogy may be carried cannot yet be determined; for neither quaternions nor symbolic logic has run its course. But even if we admit that quaternions can never realize the early hopes which were entertained of it, we cannot be blind to the fact that now, with its consort of grassmannian origin, it stands at the head of a large family of investigations which has allianced itself with the theories of groups and of matrices and which through vector analysis has actually found employment in the mathematical encyclopedia. It would perhaps be rash to forecast the future of symbolic logic by any too close translation of what has occurred in another field; but the attempt to look somewhat into the future has an irresistible lure. To get on safer ground and to reach a point from which the setting of the three books under review may be seen to better advantage, it will be well to run briefly through the history of symbolic logic.

The rudiments of a symbolic method in logic may be seen in Euler's diagrams. For the algebra of classes or of propositional functions they offer a ready expression of the relation of inclusion, and the inequality of mathematics is so vividly suggested that no surprise should be felt if the discovery were made that Euler had experimented with truly symbolic methods. Certain it is that Leibniz perceived the possibility and apparently believed in the desirability of the introduction of symbolism into logical doctrine. It was Boole, however, who first made public an extended treatment of logic in symbolic terms. It should not be overlooked that his point of view was to treat logic by mathematical methods; not mathematics by purely logical methods.

Following Boole, there was a considerable body of work done by those primarily interested in logic. We may mention Charles Peirce, Mrs. Ladd-Franklin, MacColl, and Schröder. Aside from affording a formulation of the simpler laws and processes of logical procedure, these investigations disclosed the possi-

bility of constructing a boundless extension of formal or algebraic logic by mere manipulation of the formulas. In fact it was possible to go on and on in logical algebra just as in ordinary algebra. The huge structure thus raised in all its intricacies and complications must naturally have an even less vital bearing on the fundamental problems of logic than similar refinements of pure mathematics have in the discussion of the physical world. There would be the same fruitfulness and the same barrenness.

It was at this point that Frege and Peano came forward with essentially new ideas — Frege in the early eighties and Peano in the late eighties of the last century. Unfortunately Frege's highly complicated symbolism prevented his keen philosophical reasoning from receiving much attention until very recently, and Peano's symbolism would very likely have proved a similar obstacle had his methods not appealed so strongly to his pupils as to build up a large school whose combined results naturally forced themselves on public attention. The new ideas were simply that mathematics was but a branch of logic, of symbolic logic. As early as 1889 Peano was very insistent on the fact that numbers and points were mere symbols. This was a great step to make.

And yet, when once made, it could hardly fail to appear inevitably suggested by what had already been accomplished. The well known geometries of Lobachevsky and Riemann were evidence enough that it was not our physical space but the system of axioms which afforded the logical foundation of geometry. And the geometric principles of correspondence, whether the simple dualities in plane and space or the more complicated transformations such as the line-sphere transformation of Lie, showed conclusively that it was not the undefined symbols and axioms but their interpretation which made the visual geometry. Thus on both sides the material for making the generalization lay right at hand. This new point of view brought to symbolic logic a new vitality, much as the discovery of a new relation between mathematics and physics has always brought fresh energy to mathematical investigations. The early point of view of Boole had now been changed to its opposite.

This change was far more vital than might at first be imagined. It emancipated logic from the typical form of reasoning found in the statements: All liberals are free-traders; X is a

liberal; hence X is a free-trader. If logic is to handle all mathematics, it must deal successfully with the null class and the infinite class. The older logic was too vague, too full of tacit assumptions to stand such a strain. The researches of Cantor on transfinite numbers would alone show that the little word "all" had to be handled very gingerly. The wider field which was opened was not for symbolic logic merely, but for general logic as well. If the sufficiency of a definition or concept is to be tested, there is small value in selecting the simplest case as a test stone; the most complicated should be sought out, and Cantor's work has furnished the necessary complications for considerably clarifying, enlarging, and sharpening up our fundamental logical assumptions. The process is not yet finished.

Some of the advance which has been made towards a more perfect logic has been done without the use of what is strictly symbolic logic. The time may come when symbolism may be entirely discarded for such purposes and when symbolic logic as such may be a thing of the past. It should not then be forgotten, however, that those who first started upon the advance introduced an improved symbolic logic to help their progress, which without it would have been much slower and less effective. At present there remains much to accomplish which apparently cannot be done effectively without some acquaintance with symbolic methods. But it is not easy to admit that such a complete use of formalism as is advocated and illustrated by Peano in his Formulario can ever reach general acceptance. If it could, it may be true that a volume the size of the Jahrbuch might contain practically all the mathematical investigation of a year instead of merely short abstracts. But it is a recognized psychological fact that although condensation and abstraction up to a certain point aid the mind in grasping a result, yet after that point they render the task more difficult on account of the fatigue incident to excessive attention. The problem is essentially one in minimum fatigue, and notwithstanding the experiments on mental and physical dieting we shall probably as a majority stick to the feeling that we all need some hay with our grain.

Couturat's Algebra of logic aims to supply in brief and readily accessible form the elementary principles of the subject developed from the algebraic point of view rather than from

the logical. It thus gives the mathematician an example of a non-quantitative algebra and the logician a chance to become acquainted with formal methods under auspicious circumstances. The terms and symbols are carefully explained and illustrated, and the algebraic work is for the most part maintained on a very simple and elementary level. The essentials only are given. For the learner this is a great advantage as compared with the detailed exposition of a large treatise like Schröder's. The logician will doubtless regret that the author did not give an article or two to the discussion of the different canonical types of syllogism, and indeed it might have been well to put in the text at an early point the elegant inconsistent triad of Mrs. Ladd-Franklin with a few remarks on its application. As it is, the author remains true to his intention of presenting the algebra of the subject without especial reference to the classical logic.

The first twenty articles give the reader a statement of the postulates and their immediate consequences. The matter is so presented as to admit of the dual interpretation conceptual and propositional, and these interpretations are constantly formulated in words. The postulates are chosen from the point of view which appears to the author as that most nearly in coincidence with the usual logical interpretation, and are not subjected to the critical examination which would be necessary if the chief interest were to set up a complete system of independent postulates. This is, of course, very much to the advantage of the ordinary reader who would merely be wearied by such refinements. On the other hand the author takes pains to make such comments as will inform the careful reader of many of the refinements which are really essential to a correct point of view. For instance on page 15, after showing how inclusions may be transformed into equalities, he states that if the relation of equality instead of the relation of inclusion had been taken as an undefined symbol, then the relation of inclusion could have been defined and the principle of the syllogism could have been proved. This statement contains the meat of what is perhaps the most fundamental idea in the whole subject, namely, that the definitions and theorems depend on the undefined symbols and postulates, which any author may assume with considerable arbitrariness, so that what is a theorem or definition for one author may be an undefined symbol or postulate for another, and vice versa. The presentation of the troublesome 0 and 1

(all or true) and of negation is such as to leave little room for doubt or misunderstanding. In explaining the law of duality, the distinction between primary and secondary propositions is insisted upon.

Articles 21 to 55 inclusive are of a different nature; they deal more especially with the logical algebra and contain numerous formulas which do not admit such an immediate logical interpretation as those of the preceding articles. The development of logical functions, the solution of equations in one or more unknowns, and the problem of elimination are among the subjects which might appeal most strongly to the mathematicians as affording differences and analogies to the corresponding problems of ordinary algebra. The laws of consequences and causes and the logical machine of Jevons would very likely be more of interest to the logician proper. Throughout these developments the author keeps the manipulation of the symbolism as simple as possible; but undoubtedly it would still offer obstacles to those quite unused to algebraic methods.

The concluding articles of the book are given over to the discussion of formulas which arise only in the calculus of propositions. As is known, it is only to a certain extent that the duality between propositions and classes may be carried. The principle of assertion introduces a fundamental difference. The postulate $(a = 1) = a$, which expresses this principle, enables a considerable reduction in the general formulas to be carried out — in short the calculus of propositions reduces to the calculus of classes where the universe of discourse has but one element.

In conclusion Couturat makes the point that this algebra of logic which he has set forth is of the nature of a formalization and extension of classical logic without going on to questions essentially different from those which have been treated since the time of Aristotle, namely, the relations of inclusion and implication, whereas a more general theory of relations seems necessary to a complete symbolic logic which shall suffice for the purposes of all mathematics.

From what has been said it will undoubtedly appear clear that Couturat has supplied a book which in no way duplicates anything in logical literature and which furnishes an easy and thorough introduction to the larger treatises. At the same time it is sufficiently advanced to cover more than the amount of symbolic logic which may be required by one not primarily interested in the subject. The author has had on hand for

some time the preparation of a general treatise on logistics in which we may expect to find a completer discussion of symbolic logic as it now is.

For thirty-five years MacColl's name has been associated with the advance and with the criticism of symbolic logic. Aside from his numerous publications in various periodicals, he has given a somewhat lengthy account of his system in the third volume of the reports of the congress of philosophy at Paris in 1900. His present publication in book form appears to be an amplification of that account. One of his chief aims, as he states, has been to adopt notations which shall lead to a simple discussion of problems of probability and limits. It should be remarked that he uses limit in the sense in which it is used in the expression "limits of an integral" and not as a constant approached by a variable. For his purposes MacColl has seen fit to introduce a notation entirely different from any we have seen elsewhere. There is no suggestion, so far as the mere symbolism goes, of the relations of inclusion, whether subsumption or implication. A implies B is written $A:B$ and the statement that if A belongs to class B then C belongs to class D is formulated as $A^B:C^D$.

It may be interesting to go somewhat into detail as to the author's notation. He has five general exponents, $\tau, \iota, \epsilon, \eta, \theta$, which are used respectively to denote that the proposition to which they may be affixed is true, false, certain, impossible, variable. The difference between true and certain is this: A^τ means that A is true in some particular instance, whereas A^ϵ means that it is always true, that is, of probability 1. Again A^ι means that A is false in some particular instance, whereas A^η means that A contradicts some datum or definition, that its probability is 0. To quote further we find that "A^θ asserts that A is neither impossible nor certain, that is, that A is possible but uncertain. In other words, A^θ asserts that the probability of A is neither 0 nor 1, but some proper fraction between the two." As the author apparently does not define probability our interpretation of what he means by it must belong to class θ.

According to the generally accepted definition, we may agree that the probability of a certainty is 1 and of an impossibility is 0; but unless the class of objects is finite we cannot agree that the statements "possible but uncertain" and "having a

probability which is a proper fraction" are identical. For suppose we mark off an interval of length unity on a line and let us further mark the middle point of the interval with the letter M and let P represent any point of the interval. Then the proposition P is M appears to be neither certain nor impossible but variable and so with the proposition P is not M. Yet in the former case the probability would appear to be 0 and in the latter case to be 1. Whether the author has been insufficiently careful in his statement concerning A^0 or not, the fact remains that probability is a dangerous thing and might better be left out. As a mathematician, one would hesitate a long while before he was willing to commit himself to a definition of probability which could be applied to a proposition derived from a propositional function which had an infinite number of applications in the universe of discourse.

The first six chapters of MacColl's book are used to set up his logical system and develop his analysis. It would take an expert a long time to decide which of the many statements the author intends to have regarded as postulates or axioms. It is probable that his method of procedure is not all in sympathy with the recent postulational tendency and that he reserves the right to introduce any statement at any time without making much ado over it, provided only that it appears to him to be correct. In this he is quite in touch with the everyday logician. Yet on the whole he is explicit in his statements and not readily misunderstood; it is sometimes difficult to understand him at all. For instance, there is the long standing dispute between him and some other logicians as to whether a proposition may be sometimes true and sometimes false. This seems to be largely a matter of definition. As MacColl's definition of proposition does not differ much from the usual definition of propositional function, we may be justified in concluding with some of his milder opponents that according to his own definitions he is right.

Chapters VII to XIII inclusive treat the usual subjects of classical logic. The treatment is careful and critical and affords a large amount of material for any one interested in the logical side of symbolic logic. On page 47 is found the startling statement that not one syllogism of the traditional logic is valid in the form in which it is usually presented in our textbooks. The author merely wishes to emphasize the fact that "Every A is B; every B is C; therefore every A is C" is incorrect

whereas "If every A is B and if every B is C, then every A is C" is correct. And he is perfectly right in emphasizing it. The logician might attempt to wriggle out of the difficulty in one of two ways: 1° by saying that he meant the if's and the then instead of the assertions and the therefore, or 2° by claiming that it was always assumed that the asserted premises were true. In the former case we could certainly reply that it was his business to say what he meant, and in the latter case we should point out that the useful principle of reductio ad absurdum depends on drawing valid inferences from hypothetical or else erroneous premises.

The author next takes up the nineteen classical syllogisms as revised by the insertion of if's and then's. Four of these types, namely, Darapti, Felapton, Bramantip, Fesapo, have generally been held by symbolic logicians to be incorrect forms of inference or impure syllogisms. It is certain that they do not satisfy the inconsistent triad of Mrs. Ladd-Franklin and that they are the only ones that do not. Darapti reads as follows: If every Y is Z and every Y is X, then some X is Z. The difficulty with the triad in this case is that if no Y exists there is no way of connecting X and Z, and most symbolic logicians have been willing to add $Y \neq 0$ to make the inference sure.* Of course if it be admitted that every Y implies the existence of at least one Y or that some X does not imply the existence of any X, there is no difficulty. There remains therefore only the case where Y is null and where some is taken to imply existence. MacColl claims that the inference is sure in this case; and a large number of logicians would probably agree with him. Perhaps they are right; but part of MacColl's reasoning looks suspiciously as if it could be applied equally well to cases which everybody would admit were false.

After offering so much in disapprobation of the book under review, it is a great pleasure to be able heartily to approve of the manner in which the author treats the so-called canons of syllogistic validity. These are: 1) Every syllogism has three and only three terms. 2) Every syllogism consists of only three propositions. 3) The middle term must be distributed

* It may be of interest to quote Schroeder's example. We omit the if's because the premises are so stated as to be incontestable. Every rectangular equilateral triangle is rectangular; every rectangular equilateral triangle is equilateral; hence some rectangular triangle is equilateral. The conclusion is true in spherical geometry where there are equilateral rectangular triangles but untrue in the plane where there are none.

at least once in the premises; and it must not be ambiguous.* 4) No term must be distributed in the conclusion unless it is also distributed in one of the premises. 5) We can infer nothing from two negative premises. 6) If one premise be negative, the conclusion must be so also; and conversely. The author proceeds to demolish this structure. Let us take a simple example in connection with the fifth canon. Suppose the premises: No X is Y, no Y is Z (the if's as usual are omitted). Can nothing be inferred? It is unnecessary to wake up the old-school logician for such a matter. Canon 5) is obviously sufficient. Suppose by an extremely artificial rearrangement of language we write: All Y is not-X, all Y is not-Z. Can nothing be inferred? A foolish question: this is obviously the valid Darapti and some not-X is not-Y!

It is by such chicanery as this that the logicians of the old school persist in saving the face of their canons and at the same time render the study of their logic distasteful to many and disgusting to those who are used to drawing conclusions rapidly and accurately. The author is quite justified in maintaining that his formulas are more reasonable. If one is willing to discard the four disputed cases, the inconsistent triad is even more convenient. The statements some X is Y and some Y is X are obviously equivalent and may be written $XY \neq 0$. The statements no X is Y and no Y is X are written $XY = 0$. The statement all X is Y is equivalent to no X is not-Y and is written $XY' = 0$. The triad may then be written

$(XY = 0)(Y'Z = 0)(XZ \neq 0)$ is impossible,

or $(XY = 0)(Y'Z = 0)$ implies $(XZ = 0)$, universal syllogism,

or $(XY = 0)(XZ \neq 0)$ implies $(Y'Z \neq 0)$, particular syllogism.

It takes less time to acquire a ready working knowledge of this rule than to learn the poem about Barbara, to say nothing of learning how to apply it.

The author next goes on to the question of supplying the missing premise in an enthymeme and to similar questions. He then takes up some disputed points such as the alleged duality between classes and inclusion on the one hand and propositions and implication on the other. MacColl was one

* It is extraordinary that anybody should think it worth while to canonize this latter statement!

of the first to argue that the duality is by no means complete, and by this time his contention is pretty well established despite the fact that he himself is not careful in drawing a distinction between propositions and propositional functions. Another matter in which the author has always been very active is the discussion of the null class and the paradoxes which arise from its introduction into the logical system. That there are difficulties in the theory of the null class and that these difficulties, and those connected with the class of all, perhaps more than anything else stand in the way of the ordinary logicians when they attempt to follow the symbolists no one will deny. The solution which the author gives by introducing a theory of unrealities does not, however, appear any more satisfactory to most symbolists than their theory appears to him.

Two very useful chapters are those on the solutions of some questions taken from recent examination papers and on the definitions of technical terms. One of the difficulties which a reader who is not especially trained in the nomenclature of logic experiences is the sudden appearance of some term such as modus tollendo ponens or complex constructive dilemma, and so on. The author's systematic tabulation of definitions of technical terms is therefore a great convenience. It is impossible, however, to pass over one of the author's definitions without a word of protest. He says: "Much confusion of ideas is caused by the fact that each of these words (infinite and infinitesimal) is used in different senses by mathematicians. Hence arise most of the strange and inadmissible paradoxes of the various non-euclidean geometries. To avoid all ambiguities I will define. . . . The symbol α denotes any positive quantity or ratio too large to be expressible in any recognized notation and any such ratio is called a positive infinity." The first sentence is correct although the confusion as far as mathematicians is concerned is not insuperably great. The second, about non-euclidean geometries, appears to be a bit of news. The third is apparently both ambiguous and unintelligible. Is the α beyond the alephs and omegas? As a matter of fact the α does not appear to be anywhere in particular. And this was in 1906.

The last five chapters, XIV to XVIII, deal with the calculus of limits. It is this portion of the work which the author might expect to find most fully reviewed here. But the mention will only be short. The problem is to find those values of the variable for which certain expressions remain positive or

become zero or remain negative, and more especially to determining the limits of integration for multiple integrals. The question of the limits of integrals was treated some long time since by the author in the *Proceedings of the London Mathematical Society*. He has also contributed other mathematical articles to that periodical and to the Mathematical questions and solutions from the *Educational Times*. The great advantage of his method is, as he states, that it is independent of diagrams and is equally applicable when the number of variables is great, whereas diagrams become complicated in three dimensions and impossible in a higher number. But like the old-school logicians, most mathematicians have an obstinate fondness for their own methods and will probably struggle on with them the best they can.

As regards the numerous points still in dispute between the author and other logicians one further word should be said. The disputations have been largely controversial. In some cases MacColl has won and in some he has lost; and a similar division of the spoils is not unlikely in the future. The unfortunate thing is that there has been so much controversy and so little proof whether on one side or on the other. In fact the mass of verbiage and illustration is often quite bewildering; so that the reader of the communications pro and con is unable to decide on anything. If in such cases as these the disputants would only refrain as much as possible from talking and devote their energies to setting up sets of postulates as definitely as they could and then developing their consequences with reference to alternative systems, some distinct and intelligible results might be obtained in short order. These results might be largely negative in that they showed that different logical systems led to similar conclusions with regard to all points not in dispute and to different conclusions with respects to points in dispute; but even that much conclusively shown would amount to a great deal in clearing the atmosphere. In other words why, pray, do not logicians use to a greater extent the refinements of deductive logic when they are discussing delicate points; mathematicians have adopted that point of view in dealing with geometries. Evidently the doctors do not take their own medicine.

Shearman, if his text be taken in evidence, is precisely one of these controversial and dialectic logicians. His book,

being in itself a history and a criticism of symbolic logic, is particularly hard to criticize in detail. He has much more of interest to offer to logicians, and especially to such as stand aloof from symbolic logic, than to mathematicians. For instance, he starts his introduction with the statement that it is his object to show that during the last fifty years there has been a definite advance made in symbolic logic. As there was very little symbolic logic in 1856 no one will dispute his contention, and as far as mathematicians are concerned one may say that the chief advance in symbolic logic has been made in the last twenty years. And this work is left almost untouched by the author. As his point of view is partly historical and partly critical, and as there is no very sharp distinction made between the two in the text, it is at times practically impossible to decide whether any particular series of statements is to be interpreted as a quotation from past reasoning which led to certain conclusions or as an argument of the author's.

The first chapter deals with the question whether letters should be taken to represent classes or propositions. After thirteen or fourteen pages of dialectic we reach this conclusion: "I think, then, that it is correct to hold that our letters may represent either classes or propositions, but that we must be careful to notice that the rules to be adopted in working out problems are not the same in the two cases. This view has been adopted by Boole, Venn, and Schröder, who, so far as the point in question is concerned, are thus seen to have been proceeding in the right direction." This is typical of the author's critical-historical conclusions, and the dialectic which goes before is typical of his analysis. To one at all accustomed to the use of symbols it might seem obvious that letters may be used to represent either classes or propositions or anything else, provided that the proper rules were in each case observed; but the author has seen fit to hedge a bit in a footnote, where he states that his argument has reference to the earlier problems of symbolic logic and that "when we come to deal with problems that are not included within the scope of the Boolian treatment, I admit that it is better to let symbols stand primarily for propositions."

This sort of elaborate treatment of questions of notation, this hedging about every statement with a qualification, this mixture of criticism and history to a point where the reader cannot tell what is meant, this formulating of conclusions in the first person

with the author's own position as a final test for correctness and rightness abound throughout the book. They are not peculiar to Shearman alone, but permeate the work of a large number of logicians who apply the adjective symbolic to their work, although they rarely if ever do the smallest piece of research with truly symbolic methods. What real justification they have in applying the term to themselves is merely a matter of conjecture. Perhaps they do not know what other term to use. They certainly are not classical logicians, because they have been so bold as to see that reading Attic Greek is not a sufficient condition for becoming a complete logician, and they are certainly not up-to-date symbolic logicians, because they have not perceived that reading modern Italian is a necessary condition. Notwithstanding, they attack each other with ardor, assault a really conscientious symbolist like MacColl or Mrs. Ladd-Franklin with virulence, and freely distribute their willing or unwilling applause to such authorities as Boole, Schroeder, Peano, Frege, Couturat, and Russell. Although they are thus proficient in controversy, it may be doubted if they have any even embryonic notion of what really constitutes a proof, a definition, or a postulate.

Shearman's second chapter is on the symbols of operation. He states on page 34 that most writers on the subject of symbolic logic have undoubtedly introduced symbols of operation. It is interesting to note the implication that one may perfectly feasibly construct a symbolic logic without any symbols of operation. This is but corroborative of our statement that many logicians who term themselves symbolic eschew symbolism to a very great extent. It would appear that they might better use the adjective formal in place of symbolic. For any logic which is other than an investigation in the psychology of the mental processes of reaching conclusions (as Wundt's Logik, for instance) is bound to be symbolic to a certain extent. The mere employment of language at all is symbolism. To go a little further the use of X's and Y's and Z's in stating the syllogisms is symbolic. To this extent all formal logic is symbolic. But the real essence of symbolic logic as contrasted with merely formal logic is its recognition of the value of a preponderating use of symbols as against language of the ordinary sort on account of the greater precision, the greater abstraction, and the readier manipulation which are thereby obtained.

Although a large amount of the author's second chapter will

appear either trite or misguided to those acquainted with symbolism, it may be of value to others; and some of the discussions, such as that connected with the difference between exclusive and non-exclusive alternatives, are illuminating to a larger circle of readers. The next chapter on the process of solution gives an idea of solution whether analytic or diagrammatic. The presentation would gain considerably in clearness if the author had inserted at least one diagram and had outlined the analytic method with more analytic detail. The next chapter discusses the question of the extensive versus the intensive basis for the calculus of logic. The treatment is mainly carried on as a critique of Castillon's system, which was based on intension. This is particularly interesting inasmuch as Castillon's logic is comparatively little known. The author inclines very strongly to the opinion that the basis should be extensive — in fact he is by no means completely in sympathy with Russell's idea that it is in a domain intermediate between pure intension and pure extension that symbolic logic has its lair. But just what the author would do in handling the infinite classes does not appear. He avoids the very subject which has been mainly responsible for the recent advances in symbolic logic.

Shearman spends chapter V in trying to show that Jevons and MacColl have not contributed much to symbolic logic. Jevons's deficiency was due to his lack "in power of originating important logical generalizations and his failure to appreciate the full significance of the work done by other logicians." This is perhaps true; but the statement is so extreme that it would apply with equal or greater force to so many others as to nullify its value as a criticism of Jevons in particular. MacColl's difficulty is his lack of cooperation with other symbolists. The words about Jevons are few; but more than twenty pages are devoted to reducing MacColl to an infinitesimal. The length of the argument shows one thing very conclusively, namely, MacColl's prominent position in symbolic logic during the last thirty years. As has been stated above, the points in dispute with MacColl are partly matters of definition in which any author is more or less free to use his own discretion, partly matters of notation in which an author is bound by little more than his regard for the ease of his readers, partly matters connected with much mooted questions which are not yet settled entirely to the satisfaction of any one class of logicians, to say nothing of all classes together.

Until this triplicate division of the dispute is more carefully observed and until more refinements have been employed than are available in a mere argument, we cannot agree with the author in stating that the controversies have been settled, even if a good deal of light has been thrown on them.

Chapter VI is on the later logical doctrines, namely, the doctrine of multiple quantification, the logic of relatives, and the new symbolic logic. The first will be passed without comment. In treating the logic of relatives, the author appears to be unacquainted with Russell's or Royce's investigations on the subject, and without this any present discussion of relatives is to a large extent lacking in completeness. This defect is partly remedied in the treatment of the new symbolic logic. Here there is some account of the work of Frege, Peano, Russell, and Whitehead. This will probably afford interesting reading for logicians rather than for mathematicians, who will prefer to go to the original sources instead of relying on what is necessarily an inadequate account. In the concluding chapter on the utility of symbolic logic, the author lapses into a metaphysics or something similar that apparently does not advance him very far toward showing that symbolic logic has any particular utility.

In concluding with this critical-historical study of the logical calculus, it should be stated that the mathematician must not turn to it in expectation of finding what he is most likely to want. There is nothing in it which will throw light on the nature of a defintion, a postulate, or a proof, or of what constitutes a deductive system; little or nothing which will suggest the existence of reasoning other than syllogistic; and not much more which would indicate the analytic elegance of the threefold calculus of propositions, classes, and relations. The mathematician is almost certain to find more of value to him in the excellent though brief account of symbolic logic by E. V. Huntington and Mrs. Ladd-Franklin in the ninth volume of the Encyclopedia Americana. It would be difficult to cite a better reference.* Logicians may find in Shearman's book much that is interesting and valuable to them; but as a mathematician writing in a mathematical journal, it is impossible to

* We should not omit to cite also the admirable and quite recent brochure entitled Uno sguardo al nuovo indirizzo logico-matematico delle scienze deduttive; discorso letto dal professore M. Pieri inaugurandosi l'anno accademico 1906–'07 nella Reale Università di Catania. Francesco Galati, Catania, 1907. 62 pp.

refrain from cautioning them against imagining that they are becoming acquainted with symbolic logic, or the deductive system in general, as mathematicians know it and use it. It may be hoped that logicians, too, will see fit to consult the article in the encyclopedia just mentioned or some other source of similar character, such as Pieri's inaugural address, before they permit themselves to form an opinion on the accomplishments, value, and recent advances of symbolic logic.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS.

SHORTER NOTICES.

H. Durège, Elemente der Theorie der Funktionen einer komplexen veränderlichen Grösse, in fünfter Auflage, neu bearbeitet, von LUDWIG MAURER, mit 41 Figuren im Text. Leipzig, B. G. Teubner, 1906. 397 + x pp.

THE present work, although styled the fifth edition of Durège's well-known "Elements," is in reality a new treatise. The title and the short historical introduction have been retained; aside from these, we have not remarked a trace of the original work. Nor could it be otherwise. The theory of functions has grown enormously since the days of Riemann. On the one hand, new fields have been opened up and explored, on the other the old tools of research have been given a greater refinement and many new ones have been added. The present author, in preparing a new edition, quite rightly decided not to patch up the old edifice, but to tear it down completely and erect a new one in harmony with the needs and tendencies of the present day. The result is an up-to-date treatise of moderate proportions, clearly and attractively written, which will surely have a widespread and well-deserved popularity.

The book starts out with an introductory chapter on real variables. Dedekind's theory of irrational numbers is sketched; such notions as simple and multiple limits, upper and lower limits, uniform convergence, also a few notions from the theory of point aggregates are briefly treated. The subject of integration is developed more fully and terminates with Gauss's relation between line and double integrals, which is later used to prove Cauchy's fundamental theorem.

After a short chapter on the arithmetic of complex numbers and their geometric representation, we begin on page 75 the study of the theory of functions of a complex variable. The definition of an analytic function chosen by the author, is Cauchy's, viz.: $f(z)$ shall have a continuous derivative. One hundred pages, divided into three chapters, suffice for the author to give the reader a tolerably well rounded survey of the general theory of one valued functions. The selection of topics is felicitous, the treatment is fresh and individual, and finally a happy mean is held between laxity and extreme rigor in the proofs.

The author has given Cauchy's integral relation

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z}$$

a use we have not noticed elsewhere. He shows very readily that if f is a continuous function of x, y and satisfies this relation, it is analytic. By the aid of this simple property, he shows that a uniformly convergent series or product of analytic functions is analytic. In this connection the reviewer may be permitted to express his surprise that Herr Maurer has not made use of the theorem of page 118 relative to the termwise integration of a series, to shorten very much the proof of Taylor's and Laurent's theorems. With its aid the lengthy discussion of the remainders is unnecessary.

Now follows a chapter on doubly periodic functions. This break in the general theory serves a two-fold purpose. First, it gives the reader a chance to rest after a rather arduous march; secondly, he is given an opportunity to apply his new instruments to the study of an interesting and important class of functions. We know there is nothing to sustain a healthy interest like doing things.

We may be allowed to call attention to the pretty way the $\wp$ function is introduced. As simplest elliptic function, we choose one of order two. We take its pole double, as the order of its derivative is then three instead of four. For convenience we suppose this pole to be at the origin. The development of the function is thus

$$\frac{a_{-2}}{z^2} + a_0 + a_1 z + a_2 z^2 + \cdots$$

The choice of a_{-2} and a_0 completely fixes the function. We take now $a_{-2} = 1$ and $a_0 = 0$. It is then easy to show that this function is Weierstrass's $\wp$ function. Let us note in passing that the author has broken away from the traditional symbol $\wp$, which he replaces by p. The reviewer has never been able to understand the vogue of this fetish, unless it be that mathematicians, like children, take pleasure occasionally in some strange and outlandish novelty.

Leaving the elliptic functions, the reader is introduced to the theory of many valued functions. Here the notions analytic continuation, Riemann surfaces and Schwarz's principle of reflection are briefly treated. As an instructive example, the representation afforded by the elliptic function is studied in detail.

A noteworthy chapter on algebraic functions now follows. It would be difficult to imagine a better choice of material than that which compactly fills its fifty pages. Puiseux's cycles are determined and the corresponding Riemann surface is constructed. No attempt is made to reduce the general surface to one simply connected; the author has wisely restricted the discussion to the all important case where its branch points are of the first order. The chapter closes with a study of functions which are one valued on a given Riemann surface.

The last chapter, occupying just 100 pages, is devoted to the theory of homogeneous linear differential equations of the second order. This is an innovation in text-books on the function theory. We believe, however, it is one which will win many friends for the book.

The first problem is to determine the form of the solution in the vicinity of its singular points. The presence of certain singular points called points of indetermination causes great complication. Criteria are therefore established which characterize equations free from such points. They are the equations first studied by Fuchs and have been named in his honor. Simple examples of these are the equations satisfied by the hypergeometric function and the P function of Riemann. These are now studied in detail. When the independent variable describes a circuit about a branch point the fundamental integrals are subjected to a linear substitution. The coefficients of the latter are now determined, and the result finds at once an application in the study with Riemann and Schwarz of the representation afforded by the quotient of two independent solutions of

the hypergeometric equation. We are thus led directly to the theory of automorphic functions. Space is found for a short discussion of the dreieck functions, *i. e.*, functions which map the half plane on a triangle whose sides are circles.

We hardly need to add in closing that we recommend Herr Maurer's book most heartily to the student. Even the instructor will, we doubt not, find fresh inspiration in perusing its pages, and find here and there a mode of treatment which he will be tempted to incorporate into his own lectures.

JAMES PIERPONT.

Elliptische Funktionen. Von HEINRICH BURKHARDT. Zweite, durchgesehene und verbesserte Auflage mit zahlreichen Figuren im Text. Leipzig, Veit and Company, 1906. 373 + xvi pp.

THE present edition is essentially a reproduction of the first, except that here and there a proof has been improved or a typographical error has been corrected. Numerous friends of the book have sent the author lists of errata. Thanks to their cooperation, the author hopes the formulas are now entirely reliable. This is certainly a most important feature in a subject which almost suffers from its inexhaustible wealth of formulas.

For a detailed account of the contents and tendencies of this superior work, the reader may consult an extended review in this BULLETIN for July, 1900, pages 452–463.

JAMES PIERPONT.

Quadratic Forms and their Classification by Means of Invariant Factors. By T. J. I'A. BROMWICH. Cambridge University Press, 1906. viii + 100 pp.

THE theory of elementary divisors, with which this book deals, is one of the most useful and perfect of algebraic theories. Although it is now nearly forty years since Weierstrass's fundamental paper was published, no treatment of the subject appeared in English until the year 1904, when a brief discussion was included in Mathews's revision of Scott's Determinants. This treatment is far from being suited to the needs of one wishing to penetrate for the first time into the theory, and the same is true of the only treatise on the subject which exists in any language, that of Muth. The appearance of Mr. Bromwich's book is therefore to be hailed with satisfaction as affording the

first opportunity for those who prefer to use the English language to become familiar with this subject without an undue expenditure of labor.

The author replaces Weierstrass's term *elementary divisor*, which is now in universal use, by the term *invariant factor*. This is rather a high-handed proceeding and would seem to require justification. Weierstrass's term is a suitable one and admits of satisfactory translation into English; so that it is hard to see what the author's purpose can have been in making the change.

The author does not develop the theory except so far as it relates to the problem of the classification of quadratic forms; and, in fact, the whole book is directly connected with this problem, and with its geometrical interpretations in the cases of two, three, four, and to a less extent, five or six variables. The constant reference to geometrical illustrations, and the introduction of simple geometrical cases at an early stage form an admirable feature of the book. The author has elaborated a method of treatment, due in its essential features to Kronecker, which has the advantage over Weierstrass's original method of avoiding the necessity for the proof of certain difficult theorems concerning determinants.

In one respect the treatment is very complete since it covers not merely the cases originally considered by Weierstrass, but also the *singular case* first treated by Kronecker.

The method of presentation is thoroughly elementary, and a reader with some slight familiarity with the use of trilinear coördinates should have no difficulty in working through the little volume from cover to cover, and, in the end, really knowing what elementary divisors are and how they are to be used. After having said this, it may seem strange to add that the main results of the theory are not sufficiently emphasized, sometimes hardly even explicitly stated. This, however, is the case, and it makes the book of very much less value than it might otherwise have been to persons who wish to get at the main lines of the theory without going through all the details of the proofs and of the numerical examples which are more or less mixed in with them. The volume has no index.

MAXIME BÔCHER.

NOTES.

At the annual meeting of the London mathematical society held on November 14, the following officers were elected: W. Burnside, president; A. E. H. Love and J. H. Grace, secretaries; J. Larmor, treasurer; two vice-presidents and ten other members of the council. The following papers were read: By J. H. Maclagan-Wedderburn, "On hyper-complex numbers"; by T. J. I'a. Bromwich, "Addendum to a paper on the inversion of a repeated infinite integral"; by G. H. Hardy, "Generalization of a theorem in the theory of divergent series"; by W. H. Young, "Uniform and non-uniform convergence and divergence of a series and the distinction between right and left"; by J. E. Campbell, "Application of quaternions to the problem of the infinitesimal deformation of a surface"; by J. E. Wright, "Nodal cubics through eight given points"; by H. F. Baker, "Invariants of a binary quintic and the reality of its roots"; by E. W. Barnes, "On a transformation of hypergeometric series"; by M. J. M. Hill, "On a transformation of a certain hypergeometric series"; by J. E. Littlewood, "A general theorem of integral functions of order less than one half."

The following parts of the Encyklopädie der mathematischen Wissenschaften are announced by Teubner of Leipzig to be in the press, and will probably appear during the winter: II, 1, A, 12, Allgemeine Reihenentwickelungen, by H. Burkhardt; II, 2, B, 2*a*, Arithmetische Theorie der algebraischen Funktionen, by K. Hensel; B, 3, Elliptische Funktionen, by J. Harkness and W. Wirtinger; III, 2, C, 4, Specialle ebene algebraische Kurven, by G. Kohn; C, 5, Allgemeine Theorie der höheren algebraischen Flächen, by G. Castelnuovo and F. Enriques; III, 3, D, 6*b*, Andere Differentialgleichungen der Geometrie, by C. Guichard; D, 7, Berührungstransformationen, by G. Scheffers; IV, 1_1, Heft 4; IV, 1_2, Heft 2; IV, 2_1, Heft 4; IV, 2_2, Heft 2; V, 1, Heft 4; V, 2, Heft 3; VI, 2, Heft 2.

Of the French edition of the encyclopædia, I, 1_3, I, 2_1, I, 3_2 and I, 4_2 are in the press.

The following additional works are in the press of Teubner; Führer durch die mathematische Literatur für Studierende, by F. Müller; Die Erfindung der Infinitesimalrechnung, by G.

KOWALEWSKI; Mathematical papers of the far east, by Y. MIKAMI; Grassmann's Werke, volume 3; Didaktik der Mathematik, by A. HÖFLER; Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen, by N. v. DANTSCHER; Mengenlehre, II, by A. SCHOENFLIES; Theorie der algebraischen Zahlen, by K. HENSEL; Beiträge zur Theorie der linearen Transformationen, by W. SCHEIBNER; Grundzüge der Differential- und Integralrechnung, by G. KOWALEWSKI; Vorlesungen über Variationsrechnung, by O. BOLZA; Lehrbuch der Variationsrechnung, by C. CARATHÉODORY and E. ZERMELO; Algebraische Funktionen und ihre Integrale, by W. WIRTINGER; Vorlesungen über darstellende Geometrie I, by F. v. DALWIGK; Leitfaden der Kartenentwurfslehre, II, by K. ZÖPPRITZ; Vorlesungen über Geometrie, I, 1_2 (second edition), by A. CLEBSCH and F. LINDEMANN; Entwickelungen nach oszillierenden Funktionen (part six), by H. BURKHARDT; Geometrie der Kräfte, by H. E. TIMERDING.

AT the recent annual meeting of the royal society of London the royal medal was awarded to Dr. E. W. HOBSON, of Cambridge University, for his contributions to mathematics, and the Sylvester medal was awarded to Professor W. WIRTINGER, of the University of Vienna, for his researches in the theory of functions.

THE city of Faenza, Italy, in which TORRICELLI was born, will commemorate the three hundredth anniversary of his birth by the publication of his complete works. Professor J. VASSURA, of the royal lyceum at Forli, has been appointed chairman of the committee of publication.

PROFESSOR K. CARDA, of the technical school at Vienna, has been appointed to a full professorship of mathematics at the technical school at Prague.

PROFESSOR ST. JOLLES, of the technical school at Charlottenburg, has been promoted to a full professorship of mathematics.

PROFESSOR G. KOWALEWSKI, of the University of Bonn, has been appointed professor of mathematics at the academy of mines at Clausthal.

DR. M. LELIEUVRE has been appointed professor of rational mechanics at the scientific school at Rouen.

DR. G. HERGLOTZ, of the University of Göttingen, has been promoted to an associate professorship of mathematics.

DR. J. PLEMELJ, of the University of Vienna, has been appointed associate professor of mathematics at the University of Czernowitz.

DR. —. ROCQUEMONT has been appointed professor of applied mathematics at the scientific school at Rouen.

DR. —. GASSER has been appointed docent in geodesy at the technical school at Darmstadt.

DR. —. SALKOWSKI has been appointed docent in descriptive geometry at the technical school at Charlottenburg.

AT Williams College, Dr. C. S. ATCHISON and Mr. G. R. CLEMENTS have been appointed instructors in mathematics.

MR. G. E. CARSCALLEN has been appointed assistant in mathematics at the University of Illinois.

DR. GEORG SIDLER, emeritus professor of mathematics and astronomy at the University of Bern, died November 9, 1907, at the age of 77 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERRY (W. J.). Differential equations of the first species. New York, Van Nostrand, 1907. 12mo. Cloth.

BOSMANS (H.). L'algèbre de J. Peletier du Mans départie en deux livres. Bruxelles, 1907. 8vo. 61 pp.

CAIN (W.). A brief course in the calculus. New York, Van Nostrand, 1907. 12mo. 7 + 280 pp. Half leather. $1.75

CHINI (M.). Lezioni di algebra, ad uso dei licei. Vol. I. Livorno, Giusti, 1907. 16mo. 8 + 166 pp. L. 1.80

EULER. Festschrift zur Feier des 200. Geburtstages Leonhard Eulers. Herausgegeben vom Vorstande der Berliner Mathematischen Gesellschaft. Leipzig, 1907. 8vo. 137 pp. M. 5.00

FALISSE (V.). Cours de géométrie analytique plane. 6e édition. Bruxelles, 1906. 8vo. 572 pp. Fr. 7.00

GOSIEWSKI (W.). Zasady rachunken prawdopodobienstwa. (Principes du calcul des probabilités). Warszawa, 1906. 8vo. 265 pp. M. 6.00

HEUSSEL (G.). Ueber permutable Gruppenbasen aus zwei Elementen. (Diss.) Giessen, 1907. 8vo. 44 pp.

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